Functions of several variables

(M.Sc. (MATHEMATICS), Paper - VI)

(Real Analysis - II)

Lecture - 04

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Functions of two variable

Continuity :→ A function $f : D \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$ is said to be continuous at a point (a,b) if

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$$

$\boxed{\text{OR}}$

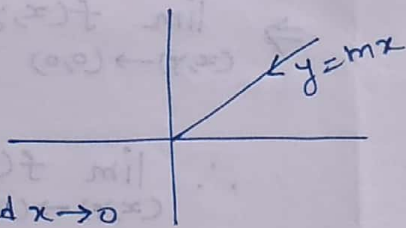
A function $f : D \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$ is said to be continuous at a point $(a,b) \in D$, if for any $\epsilon > 0$, $\exists$ a neighbourhood of N of (a,b) such that

$$|f(x,y) - f(a,b)| < \epsilon \quad \forall\ (a,b) \in N.$$

Example ① Investigate the continuity at $(0,0)$ of

$$f(x,y) = \begin{cases} \dfrac{x^2-y^2}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

Solution :→



We reach $(x,y) \to (0,0)$ along the path $y = mx$ and $x \to 0$

$$\therefore \lim_{\substack{y=mx \\ x\to 0}} f(x,y) = \lim_{x\to 0} \frac{x^2 - m^2x^2}{x^2 + m^2x^2}$$

$$= \lim_{x\to 0} \frac{1-m^2}{1+m^2} = \frac{1-m^2}{1+m^2}$$

Which is different for different values of m.

$\therefore \lim_{(x,y)\to(0,0)} f(x,y)$ doesnot exists $\Rightarrow f(x,y)$ is not continuous at $(0,0)$.

Example ② Show that the function

$$f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

is continuous at $(0,0)$.

Solution :→ Let $x = r\cos\theta$, $y = r\sin\theta$

$$\therefore f(x,y) = \frac{xy}{\sqrt{x^2+y^2}} = \frac{r^2\cos\theta\sin\theta}{r}$$

$$\therefore |f(x,y) - 0| = |r\cos\theta\sin\theta| = r|\cos\theta \cdot \sin\theta|$$

$$\leq r = \sqrt{x^2+y^2} < \epsilon$$

If $x^2 < \frac{\epsilon^2}{2}$ & $y^2 < \frac{\epsilon^2}{2}$

or if $|x| < \frac{\epsilon}{\sqrt{2}}$ & $|y| < \frac{\epsilon}{\sqrt{2}}$

Thus $|f(x,y) - 0| < \epsilon$, when $|x| < \frac{\epsilon}{\sqrt{2}}$ & $|y| < \frac{\epsilon}{\sqrt{2}}$

$$\Rightarrow \lim_{(x,y)\to(0,0)} f(x,y) = 0$$

$$\therefore \lim_{(x,y)\to(0,0)} f(x,y) = f(0,0)$$

Hence f is continuous at $(0,0)$. #

Partial derivatives :→ Let $f : D \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$.

The partial ~~derivatives~~ derivative of $f(x,y)$ with respect to x is generally denoted by $\frac{\partial f}{\partial x}$ or f_x or $f_x(x,y)$, while those with respect to y are denoted by $\frac{\partial f}{\partial y}$ or f_y or $f_y(x,y)$ and defined as

$$\frac{\partial f}{\partial x} = \lim_{\delta x \to 0} \frac{f(x+\delta x, y) - f(x,y)}{\delta x} \quad \text{(if exits)}$$

and $$\frac{\partial f}{\partial y} = \lim_{\delta y \to 0} \frac{f(x, y+\delta y) - f(x,y)}{\delta y} \quad \text{(if exists)}$$

Example ① If $f(x,y) = 2x^2 - xy + 2y^2$, then find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at the point $(1,2)$.

Solution :→

$$\because \frac{\partial f}{\partial x} = 4x - y$$

$$\therefore \left.\frac{\partial f}{\partial x}\right|_{(x,y)=(1,2)} = 2$$

and $$\frac{\partial f}{\partial y} = -x + 4y$$

$$\therefore \left.\frac{\partial f}{\partial y}\right|_{(x,y)=(1,2)} = 7$$

Ans.

Example ② :→ If

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

Show that both the partial derivatives exist at (0,0) but the function is not continuous thereat.

Solution :→ We approach $(x,y) \to (0,0)$ along the path $y = mx$ and $x \to 0$.

$$\therefore \lim_{\substack{y=mx \\ x\to 0}} f(x,y) = \lim_{x\to 0} \frac{mx^2}{x^2 + m^2x^2} = \frac{m}{1+m^2},$$

So, that the limit depends on the ~~above~~ value of m, i.e; on the path of approach and different for the different ~~paths~~ paths.

$\therefore \lim_{(x,y)\to(0,0)} f(x,y)$ does not exists.

Hence the function $f(x,y)$ is not continuous at (0,0).

Again,

$$\because f_x(0,0) = \lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{0}{h} = 0$$

$$\& \ f_y(0,0) = \lim_{k\to 0} \frac{f(0,0+k) - f(0,0)}{k}$$

$$= \lim_{k\to 0} \frac{0}{k} = 0$$

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